

# Einstein-Podolsky-Rosen-like correlation on a coherent-state basis and inseparability of two-mode Gaussian states

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The strange property of the Einstein-Podolsky-Rosen (EPR) correlation between two remote physical systems is a primitive object on the study of quantum entanglement. In order to understand the entanglement in canonical continuous-variable systems, a pair of the EPR-like uncertainties is an essential tool. Here, we introduce a normalized pair of the EPR-like uncertainties to discuss the role of the canonical uncertainty relation in the inseparability problem. As another physically reasonable tool, we also introduce a state overlap to a classically correlated mixture on a coherent-state basis and consider its role in the inseparability problem. The separable condition associated with the overlap determines the strength of the EPR-like correlation on the coherent-state basis in order that the state is entangled. In a standard form of two-mode Gaussian states, the separable conditions with the EPR-like uncertainties and the separable condition with the overlap to the classically correlated mixture are linked by a simple embrace relation. From this relation it is shown that the coherent-state-based condition is capable of detecting the class of two-mode Gaussian entangled states. We also consider an experimental measurement scheme for estimation of the state overlap by a heterodyne measurement and a photon detection with a feedforward operation. The parallelism between the separable condition with the state overlap and the quantum-domain condition with the Gaussian distributed coherent states is discussed associated with the standard continuous-variable quantum teleportation process. Thereby, we succeed in reconciling the condition on the channel fidelity to the condition on the resource entanglement for the teleportation including a non-unit-gain effect.

## I. INTRODUCTION

In the seminal paper [1], Einstein, Podolsky, and Rosen (EPR) considered a pair of particles, say  $A$  and  $B$ , that possesses perfect correlation not only in their positions  $x_{A(B)}$  but also in their momenta  $p_{A(B)}$ . From such a correlation, one can predict either the position or the momentum of one particle with certainty by measuring the position or the momentum of the other particle, and this seemingly contradicts the canonical uncertainty relation

$$\sqrt{\langle \Delta^2 \hat{x} \rangle \langle \Delta^2 \hat{p} \rangle} \geq ||[\hat{x}, \hat{p}]||/2 =: C, \quad (1)$$

where  $\Delta \hat{O} := \hat{O} - \langle \hat{O} \rangle$ . This type of seeming inconsistency between the quantum correlation and the canonical uncertainty relation is often termed as the EPR paradox and has been providing important aspects on foundations of quantum physics and theory of entanglement [2–5].

The EPR-type correlation is normally described by the variances of the EPR-type operators  $\hat{x}_A - \hat{x}_B$  and  $\hat{p}_A + \hat{p}_B$ , and the measured uncertainties can be a signal of quantum entanglement. Duan *et al.*, [6] have introduced the EPR-like operators  $\hat{X} := |a|\hat{x}_A - \frac{1}{a}\hat{x}_B$  and  $\hat{P}' := |a|\hat{p}_A + \frac{1}{a}\hat{p}_B$  with a real number  $a$ , and presented an inseparable condition associated with the total variance of the operators: A bipartite state is entangled if it violates the inequality

$$\langle \Delta^2 \hat{X} \rangle + \langle \Delta^2 \hat{P}' \rangle \geq 2(a^2 + \frac{1}{a^2})C. \quad (2)$$

Interestingly, this condition is conducted to determine the inseparability of two-mode Gaussian states. To be specific, for any given entangled two-mode Gaussian state, there exists a proper local Gaussian-unitary transformation and a parameter  $a$  so that the inequality of Eq. (2) is violated. Its implication is that the origin

of the inseparability of two-mode Gaussian states lies on the strength of the EPR-like correlation.

A quantum state on a bipartite system  $AB$  is called separable if its density operator can be written in the convex sum form of the products of local density operators as  $\rho_{AB} = \sum_i p_i (\rho_i)_A \otimes (\sigma_i)_B$  where  $(\rho_i)_A$  and  $(\sigma_i)_B$  are local density operators of the system  $A$  and  $B$ , respectively, and  $p_i$  is a probability distribution that satisfies  $p_i \geq 0$  and  $\sum p_i = 1$ . A quantum state is said to be entangled if it is not separable. The separable density operator preserves its positivity under the transpose of its local density matrix. This property of positive partial transposition cannot hold for many of entangled density operators, and non-positivity of the partial transposition is a signal of entanglement [7]. An important fact is that the class of Gaussian entangled states belongs to the entanglement with the non-positive partial transposition [8, 9]. It is shown that many of known separable conditions concerning the continuous-variable states, which include Eq. (2), can be derived by using partial transposition for moments of the annihilation operators and the creation operators [10, 11].

In quantum optics, the canonical variables correspond to the phase-space quadratures of optical modes, and their statistics can be measured by the homodyne measurement. This enables us to determine the moments of annihilation and creation operators in experiments. The homodyne measurement is a Gaussian-measurement tool and plays a central role in the continuous-variable quantum information processing [12]. Another important Gaussian measurement is the heterodyne (double homodyne) measurement. It measures the complex amplitude  $\alpha$  of an optical coherent state  $|\alpha\rangle$  and gives the projection probability to the coherent state  $\langle \alpha | \rho | \alpha \rangle$ . The canonical quadratures and coherent-state amplitudes provide similar phase-sensitive information of the optical modes, and both of them are thought to be useful to observe

the properties of Gaussian states. Hence, it is reasonable to consider entanglement verification methods associated with the measured correlations of the heterodyne detection. There have been several approaches to suggest the relation between heterodyne statistics and entanglement mainly associated with the transmission of coherent states [13]. It might be also insightful to consider the inseparability problems related to the phase-space distribution [14–16]. However, their implication with respect to the EPR-like correlation has little been discussed.

While the variances of the quadratures and the covariance matrix are often employed to observe the property of continuous-variable quantum states, the notions of the fidelity and the state overlap are normally employed to describe the performance of the quantum channel or quantum operation [17]. Any physical operation is considered to be a quantum channel that transforms quantum states in the completely positive and trace-preserving manner. The separability property for quantum channels is defined through the notion of the entanglement breaking. A quantum channel is called an *entanglement breaking* if its action on a subsystem of any bipartite state makes the state separable [18]. Its different definition is that the Choi-Jamiołkowski (CJ) isomorphism of the channel is separable. If the CJ isomorphism is inseparable, the quantum channel is said to be in *quantum domain*. A quantum-domain condition becomes an inseparable condition due to the isomorphism. In many of quantum information processing protocols, a central task is to transmit entanglement or to manipulate entangled signals coherently (so as not to break the entanglement). Hence, it is essential that a given physical implementation of the quantum channel is in quantum domain [19–33].

In order to verify the channel coherence via a feasible input set of coherent states, a condition for a quantum-domain channel has been formulated in [23, 24]: A quantum channel  $\mathcal{E}$  is in quantum domain if it violates the inequality

$$\int p_\lambda(\alpha) \langle \sqrt{\eta}\alpha | \mathcal{E}(|\alpha\rangle\langle\alpha|) | \sqrt{\eta}\alpha \rangle d^2\alpha \leq \frac{1+\lambda}{1+\lambda+\eta} \quad (3)$$

where the Gaussian distribution is given by

$$p_\lambda(\alpha) := \frac{\lambda}{\pi} \exp(-\lambda|\alpha|^2) \quad (4)$$

with  $\lambda > 0$ ,  $\mathcal{E}(|\alpha\rangle\langle\alpha|)$  represents the density operator of the output of the channel corresponding to an input of the coherent state  $|\alpha\rangle$ , and  $\eta > 0$  is an effective transmission of the channel. It was shown [23] that this quantum-domain condition is conducted to determine the inseparability of one-mode Gaussian channels [34, 35]. To be specific, if a one-mode Gaussian channel is in quantum domain one can find a set of proper parameters  $(\lambda, \eta)$  and a pair of one-mode Gaussian unitary transformations so as to violate the inequality of Eq. (3). When  $\eta = 1$  the violation of Eq. (3) corresponds to the famous success criterion of continuous-variable quantum teleportation and memory [19, 20].

Recently, a simple derivation of Eq. (3) from the CJ isomorphism and the partial transposition has been reported in Ref. [24]. Thereby, a separable condition

with the state overlap to the Gaussian distributed phase-conjugate pairs of coherent states was derived. It shows that separable states have to satisfy

$$\left\langle \int p_\lambda(\alpha) |\alpha\rangle\langle\alpha| \otimes |\sqrt{\eta}\alpha^*\rangle\langle\sqrt{\eta}\alpha^*| d^2\alpha \right\rangle \leq \frac{\lambda}{1+\lambda+\eta}. \quad (5)$$

Since the state overlap in the left-hand side is written in terms of the projections to the coherent states, it might be an accessible entanglement detection tool based on the statistics of the heterodyne measurement. However, its utility and significance for the separability problem have little been discussed.

In addition, the condition of Eq. (3) was originally formulated to verify the success of an experimental continuous-variable quantum teleportation process beyond the unit-gain constraint [24]. Equation (3) determines the fidelity threshold that cannot be surpassed without entanglement. Hence, it is natural to ask how this condition turns to the condition for the quantum state employed to teleport an unknown state. This is essentially the same question to ask what kind of quantum correlation is required to succeed in the quantum teleportation. It is of fundamental to investigate the relationship between entanglement of the resource EPR beams for teleportation and the teleportation fidelity for coherent states [19, 36–39]. However, the known approaches [38, 39] are limited to the Gaussian entangled resource states, and fully closed relations with an arbitrary resource state including the case of the non-unit gain ( $\eta \neq 1$ ) have not been established yet.

In this paper we investigate the properties of the overlap condition of Eq. (5) for the separability problem. We argue that the state overlap is a form of the EPR-like correlation in a coherent-state basis and that the overlap condition gives a threshold of the coherent-state-based EPR-like correlation for the inseparability. It is shown that the separable condition with the overlap and the separable condition with the EPR-like uncertainties can be formulated in parallel, and the violation of the separable conditions can be interpreted as essentially the same phenomenon to infer the EPR paradox. For the Gaussian states given in a standard form of the covariance matrix we find a simple embrace relation between the separable conditions. This relation provides a geometric proof that the overlap condition can be conducted to determine the inseparability of two-mode Gaussian states. In this regard, the overlap condition is potentially as useful as the condition with the EPR-like uncertainties. We also consider an experimental measurement scheme to detect the coherent-state-based EPR-like correlation using the heterodyne measurement, and investigate the connection between the overlap separable condition and the entanglement-breaking condition of Eq. (3) associated with the continuous-variable quantum teleportation scheme. We find a fully closed relationship between entanglement and the fidelity for the Gaussian distributed coherent states with an arbitrary resource state. It naturally includes the non-unit-gain case and the correlation required to success in quantum teleportation is shown to be the EPR-like correlation on the coherent-state basis.

This paper is organized as follows. We start with the canonical uncertainty relation and use the notion of partial transposition to derive the separable condition with

the EPR-like uncertainties. We apply the same scenario to the derivation of the overlap separable condition. In this reason, we introduce a limitation of the phase-space localization as a sort of the canonical uncertainty relation in Sec. II. We derive the separable condition with the EPR-like uncertainties and investigate its properties in Sec. III. We derive the overlap separable condition and discuss its properties in Sec. IV. Then, we investigate the embrace relation between the presented separable conditions and address the utility of the overlap condition for Gaussian states in Sec. V. We present the experimental scheme to observe the coherent-state-based EPR-like correlation with the heterodyne measurement in Sec. VI. We also discuss the link between the overlap condition and the entanglement-breaking condition associated with the standard continuous-variable quantum teleportation scheme in there. We conclude this paper in Sec. VII.

## II. UNCERTAINTY RELATION AND PHASE-SPACE LOCALIZATION

An intuitive interpretation of the canonical uncertainty relation in Eq. (1) is that the quantum state is located on the phase space with a finite volume [See FIG. 1(a)]. When the volume is measured in terms of the uncertainty product  $\sqrt{\langle \Delta^2 \hat{x} \rangle \langle \Delta^2 \hat{p} \rangle}$ , it has to be no smaller than the limit determined by the canonical commutation relation, i.e.,  $\sqrt{\langle \Delta^2 \hat{x} \rangle \langle \Delta^2 \hat{p} \rangle} \geq |[\hat{x}, \hat{p}]|/2$ . The standard deviation describes typical width of the phase-space distribution and thus the uncertainty product indicates a degree of localization of the phase-space distribution. Hence, a limitation on the degree of localization of the phase-space distribution is imposed by the canonical uncertainty relation. Here we consider a different measure of the phase-space localization and present another form of the physical limitation.

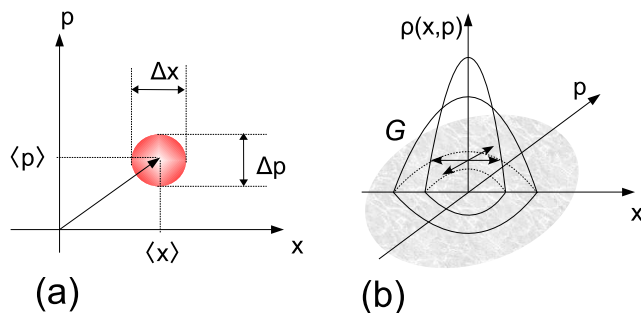


FIG. 1: (Color online) The uncertainty relation gives a limitation on the localization in the phase space. (a) A physical state is spread in the phase space so that its volume of  $\Delta x \Delta p$  is no smaller than the minimum uncertainty product of  $|[\hat{x}, \hat{p}]|/2$  due to the canonical uncertainty relation. (b) Another measure of the phase-space localization can be given by the convolution between the state distribution  $\rho(x, p)$  and a localized distribution function  $G$  on the phase space. The value of the convolution designates the concentration of the state distribution at the peak of the function  $G$ .

Let us consider the density operator of a thermal state

$$\begin{aligned} \hat{G}_\lambda &:= \int p_\lambda(\alpha) |\alpha\rangle \langle \alpha| d^2\alpha \\ &= \frac{\lambda}{1+\lambda} \sum_{n=0}^{\infty} \left( \frac{1}{1+\lambda} \right)^n |n\rangle \langle n|. \end{aligned} \quad (6)$$

Here we use the standard notation for the number state  $|n\rangle$  and the coherent state  $|\alpha\rangle = \sum_{n=0}^{\infty} e^{-|\alpha|^2/2} \alpha^n |n\rangle / \sqrt{n!}$ . The phase-space distribution of the thermal state is an isotropic Gaussian distribution and peaked at the origin of the phase space  $\alpha = 0$  [See FIG. 1(b)]. The expectation value of the thermal state  $\langle \hat{G} \rangle_\rho := \text{Tr} \hat{G} \rho = \lambda \int Q_\rho(\alpha) e^{-\lambda|\alpha|^2} d^2\alpha$  is a Gaussian convolution of the Husimi- $Q$  function  $Q_\rho(\alpha) := \langle \alpha | \rho | \alpha \rangle / \pi$ . It suggests how strong the probability distribution is concentrated around the origin. Hence, it is likely that the expectation value is maximized by the state which has a sharply peaked  $Q$  function at  $\alpha = 0$ . However, we cannot make the width of the  $Q$  function arbitrarily small, and thus  $\langle \hat{G}_\lambda \rangle$  has an upper bound. This upper bound offers another form of the physical limitation on the degree of the phase-space localization. From the second line of Eq. (6), an upper bound of  $\langle \hat{G} \rangle$  is given by the maximum eigenvalue of the thermal state  $\|\hat{G}_\lambda\|$  as

$$\langle \hat{G}_\lambda \rangle \leq \|\hat{G}_\lambda\| = \frac{\lambda}{1+\lambda}. \quad (7)$$

This relation serves as a sort of the uncertainty relation, namely, one cannot localize the physical state on the phase space so that its expectation value  $\langle \hat{G} \rangle$  surpasses the physical limit  $\|\hat{G}_\lambda\|$ . We refer to  $\langle \hat{G}_\lambda \rangle_\rho = \text{Tr} \hat{G}_\lambda \rho$  as the  $\lambda$ -localization of a density operator  $\rho$ . The equality of Eq. (7) can be achieved by the vacuum state  $|0\rangle$  and the vacuum state is the maximally  $\lambda$ -localized state. Since, the  $\lambda$ -localization is an overlap between a given state and the thermal state, it represents the probability of finding the states in the thermal distribution.

In order to see an intuitive connection between the uncertainty product and the  $\lambda$ -localization, let us consider the case where the  $Q$  function has a single peak at the origin. Let  $\delta x$  and  $\delta p$  be the width of the  $Q$  function along the real  $x$  and the imaginary  $p$  direction, respectively. Then the normalization condition  $1 = \int Q_\rho(\alpha) d^2\alpha \simeq Q_\rho(0) \delta x \delta p$  implies  $Q_\rho(0) \simeq (\delta x \delta p)^{-1}$ . Hence, for sufficiently large  $\lambda$ , we have  $\langle \hat{G}_\lambda \rangle = \lambda \int Q_\rho(\alpha) e^{-\lambda|\alpha|^2} d^2\alpha \simeq \pi Q_\rho(0) \simeq \pi (\delta x \delta p)^{-1}$ , namely, the  $\lambda$ -localization is proportional to the inverse of the uncertainty product, in a certain limit. This is a typical situation that a higher fidelity corresponds to a lower noise.

## III. SEPARABLE CONDITIONS WITH THE EPR-LIKE UNCERTAINTIES

Several inseparability conditions have been derived by using the uncertainty relations and partial transposition [40, 41]. In this section, we derive a separable condition with a normalized EPR-like uncertainty product using the partial transposition for the canonical uncertainty

relation. This separable condition is called the product condition [42, 43, 45, 46] and has a simple embrace relation to the sum separable condition of Eq. (2). In contrast to the sum condition, any point of the separable boundary of the product condition can be achieved by the product of the squeezed states. It is shown that the maximum of the EPR-like correlation can be achieved by a two-mode squeezed state (TMSS).

We start with the canonical uncertainty relation

$$\langle \Delta^2 \hat{x} \rangle \langle \Delta^2 \hat{p} \rangle \geq \left( \frac{||[\hat{x}, \hat{p}]||}{2} \right)^2 = C^2. \quad (8)$$

By introducing an ancilla system  $B$  and applying a beam-splitter transformation  $(\hat{x}_A, \hat{p}_A) \rightarrow (u\hat{x}_A - v\hat{x}_B, u\hat{p}_A - v\hat{p}_B)$  we have

$$\langle \Delta^2 (u\hat{x}_A - v\hat{x}_B) \rangle \langle \Delta^2 (u\hat{p}_A - v\hat{p}_B) \rangle \geq C^2, \quad (9)$$

where we assign the index  $A$  for the original system and assume that the real parameters for the beamsplitter  $(u, v)$  satisfy the relation  $u^2 + v^2 = 1$ . When we make the replacement  $p_B \rightarrow -p_B$  [41] we have a product separable condition [10, 45, 46]

$$\langle \Delta^2 (u\hat{x}_A - v\hat{x}_B) \rangle \langle \Delta^2 (u\hat{p}_A + v\hat{p}_B) \rangle \geq C^2. \quad (10)$$

The replacement corresponds to the transposition of the system  $B$  with respect to the number basis (see the below proof). The left-hand side of Eq. (10) is a normalized EPR-like uncertainty product so that it becomes a normal uncertainty product for the canonical variables under the partial transposition as in Eq. (9). Since the partial transposition is not a physical transformation, it is no reason to consider that Eq. (10) holds for all physical states. We can show that no separable state can violate this inequality as follows:

*Proof.* — Let us write  $\hat{X} := u\hat{x}_A - v\hat{x}_B$ ,  $\hat{P} := u\hat{p}_A - v\hat{p}_B$ , and the partial transposition, which transposes the system  $B$  with respect to the number basis,

$$\Gamma : (|l\rangle\langle m| \otimes |j\rangle\langle k|) \rightarrow (|l\rangle\langle m| \otimes |k\rangle\langle j|). \quad (11)$$

For product states, we can write the expectation value of the partial transposed observable  $\hat{O}$  as  $\langle \Gamma[\hat{O}] \rangle_{\phi \otimes \varphi} = \text{Tr}[\Gamma[\hat{O}]|\phi\rangle\langle\phi| \otimes |\varphi\rangle\langle\varphi|] = \text{Tr}[\hat{O}\Gamma(|\phi\rangle\langle\phi| \otimes |\varphi\rangle\langle\varphi|)] = \text{Tr}[\hat{O}|\phi\rangle\langle\phi| \otimes |\varphi^*\rangle\langle\varphi^*|] = \langle \hat{O} \rangle_{\phi \otimes \varphi^*}$ . Here we defined the conjugated state by  $|\varphi^*\rangle := \sum_{n=0}^{\infty} \langle \varphi|n\rangle |n\rangle$ . Since the off-diagonal elements of the position operator  $\hat{x} = \sqrt{C}(\hat{a} + \hat{a}^\dagger)$  in the number basis are real we have  $\Gamma[\hat{X}] = \hat{X}$ . In contrast, the off-diagonal elements of the momentum operator  $\hat{p} = \sqrt{C}(\hat{a} - \hat{a}^\dagger)/i$  in the number basis are pure imaginary, and we thus have  $\Gamma[\hat{P}] = u\hat{p}_A + v\hat{p}_B$ . Noting that  $\Gamma[\hat{P}^2] = (\Gamma[\hat{P}])^2$  and  $\Gamma[\hat{X}^2] = (\Gamma[\hat{X}])^2$ , we can estimate the left-hand side of Eq. (10) as  $\langle \Delta^2(\Gamma\hat{X}) \rangle_{\phi \otimes \varphi} \langle \Delta^2(\Gamma\hat{P}) \rangle_{\phi \otimes \varphi} = \langle \Delta^2\hat{X} \rangle_{\phi \otimes \varphi^*} \langle \Delta^2\hat{P} \rangle_{\phi \otimes \varphi^*} \geq \min_{\phi \otimes \varphi} \left\{ \langle \Delta^2\hat{X} \rangle_{\phi \otimes \varphi} \langle \Delta^2\hat{P} \rangle_{\phi \otimes \varphi} \right\} \geq (||[\hat{X}, \hat{P}]||/2)^2$  for any product state. Hence, for any separable state  $\rho_s =$

$\sum_i p_i |\phi_i\rangle\langle\phi_i| \otimes |\varphi_i\rangle\langle\varphi_i|$  we have

$$\begin{aligned} & \langle \Delta^2(\Gamma\hat{X}) \rangle_{\rho_s} \langle \Delta^2(\Gamma\hat{P}) \rangle_{\rho_s} \\ &= \left( \sum_i p_i \langle \Delta^2\hat{X} \rangle_{\phi_i \otimes \varphi_i^*} \right) \left( \sum_j p_j \langle \Delta^2\hat{P} \rangle_{\phi_j \otimes \varphi_j^*} \right) \\ &\geq \left( \sum_j p_j \sqrt{\langle \Delta^2\hat{X} \rangle_{\phi_j \otimes \varphi_j^*} \langle \Delta^2\hat{P} \rangle_{\phi_j \otimes \varphi_j^*}} \right)^2 \\ &\geq (||[\hat{X}, \hat{P}]||/2)^2 = C^2. \end{aligned} \quad (12)$$

From the second line to the third line, we set  $a_j = \sqrt{p_j \langle \Delta^2\hat{X} \rangle_{\phi_j \otimes \varphi_j^*}}$ ,  $b_j = \sqrt{p_j \langle \Delta^2\hat{P} \rangle_{\phi_j \otimes \varphi_j^*}}$  and use the Schwarz inequality  $|\vec{a}|^2 |\vec{b}|^2 \geq |\vec{a} \cdot \vec{b}|^2$ . ■

An important implication of the product separable condition of Eq. (10) is that the EPR-like correlation cannot be stronger than the canonical uncertainty limit without entanglement. As was shown in the proof, the EPR-like uncertainty product for a product state can be mapped into the canonical uncertainty product for its conjugated state when the EPR-like operators are normalized so that their partial transpositions form a pair of the canonical observables. This normalization might be also insightful when we recall that the canonical transformation preserves the phase-space density.

Dividing both sides of Eq. (2) by  $(a^2 + 1/a^2)$  we have a normalized sum condition

$$\langle \Delta^2 (u\hat{x}_A - v\hat{x}_B) \rangle + \langle \Delta^2 (u\hat{p}_A + v\hat{p}_B) \rangle \geq 2C, \quad (13)$$

where we set

$$(u, v) = \frac{1}{\sqrt{a^2 + a^{-2}}} (|a|, a^{-1}). \quad (14)$$

This sum condition of Eq. (13) can be also obtained by taking the square root on both sides of Eq. (10) and using the relation  $\langle \Delta^2 (u\hat{x}_A - v\hat{x}_B) \rangle + \langle \Delta^2 (u\hat{p}_A + v\hat{p}_B) \rangle \geq 2\sqrt{\langle \Delta^2 (u\hat{x}_A - v\hat{x}_B) \rangle \langle \Delta^2 (u\hat{p}_A + v\hat{p}_B) \rangle}$ . From this derivation we can see that the inequality of Eq. (10) is automatically violated if the inequality of Eq. (13) is violated. This suggests that the product condition of Eq. (10) is better to detect entanglement than the sum condition of Eq. (13). In order to show this clearly, let us write the normalized uncertainties of the EPR-like operators as

$$\begin{aligned} U &:= \langle \Delta^2 (u\hat{x}_A - v\hat{x}_B) \rangle / C \\ V &:= \langle \Delta^2 (u\hat{p}_A + v\hat{p}_B) \rangle / C. \end{aligned} \quad (15)$$

Then, the product separable condition of Eq. (10) leads to

$$UV \geq 1, \quad (16)$$

and the sum separable condition of Eq. (13) leads to

$$U + V \geq 2. \quad (17)$$

The embrace relation between Eqs. (16) and (17) is directly observed in FIG. 2. Since there is no separable state below the curve  $UV = 1$  (gray regime of FIG. 2), the states located on the boundary of Eq. (13) should



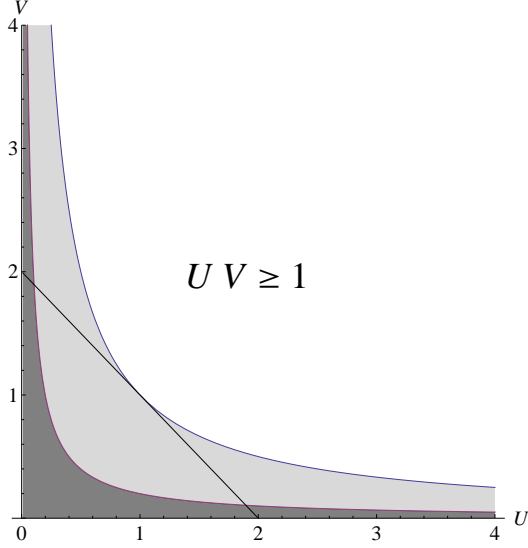


FIG. 2: (Color online) Relation between the inseparability and the normalized EPR-like uncertainties  $U$  and  $V$  in Eq. (15). The product condition of Eq. (10) [Eq. (16)] implies that any state is entangled if its location  $(U, V)$  is below the inverse proportional curve  $UV = 1$  (gray regime). On the other hand, for any point of this curve one can find a corresponding separable state (See main text). The violation of the sum condition of Eq. (13) [Eq. (17)] occurs for the states located below the straight line  $U + V = 2$ , and the entangled states detected by the sum condition belong to a subset of the entangled states detected by the product condition. The dark gray regime is physically unaccessible due to the global canonical uncertainty relation of Eq. (18). In this figure we set  $|u^2 - v^2|^2 = 0.2$  specifically so that the physical boundary is given by  $UV = 0.2$ .

be entangled except for the single point  $(U, V) = (1, 1)$ . As a result, the sum condition fails to notice the entangled states located in the area enclosed by the two curves  $UV = 1$  and  $U + V = 2$ .

It is notable that, with respect to the EPR-like uncertainties, the form of the product separable condition of Eq. (10) is the same as the familiar inverse proportional limit of the canonical uncertainty relation of Eq. (8). This implies that any point of the minimum uncertainty curve  $UV = 1$  [the equality of Eq. (16)] can be achieved by the separable states in sharp contrast to the case of the sum condition. As a separable state located on the boundary of Eq. (16), we can find the product of the squeezed states  $\hat{S}_A \otimes \hat{S}_B |0, 0\rangle_{AB}$  where  $\hat{S}$  stands for the squeezing operator. Hence, it is a signature of entanglement that the normalized EPR-like uncertainty product is smaller than the limit of the uncertainty product attainable by the product of minimum uncertainty states. This statement is consistent with the fact that any separable two-mode Gaussian state can be locally transformed into a two-mode Gaussian state whose Glauber-Sudarshan- $P$  function is positive [6, 8, 47].

Note that the physical limitation for the EPR-like uncertainty product is given by

$$\begin{aligned} & \langle \Delta^2 (u\hat{x}_A - v\hat{x}_B) \rangle \langle \Delta^2 (u\hat{p}_A + v\hat{p}_B) \rangle \\ & \geq \left( \frac{|[u\hat{x}_A - v\hat{x}_B, u\hat{p}_A + v\hat{p}_B]|}{2} \right)^2 = (u^2 - v^2)^2 C^2. \end{aligned}$$

In terms of  $U$  and  $V$ , it can be expressed as

$$UV \geq |u^2 - v^2|^2. \quad (18)$$

This inequality is saturated by the TMSS

$$|\psi_\zeta\rangle_{AB} = \sqrt{1 - |\zeta|^2} \sum_{n=0}^{\infty} \zeta^n |n\rangle_A |n\rangle_B, \quad (19)$$

with  $\zeta = v/u < 1$ . We can observe that the TMSS is located at  $(U, V) = (\sqrt{u^2 - v^2}, \sqrt{u^2 - v^2})$  on the  $U$ - $V$  plane and that the physical boundary of Eq. (18) can be covered by the state  $\hat{S}_A \otimes \hat{S}_B |\psi_\zeta\rangle_{AB}$  similar to the case that the product of the squeezed states covers the separable boundary of Eq. (16). Noting the relation  $U + V \geq 2\sqrt{UV} \geq 2\sqrt{u^2 - v^2}$ , we can see that the physically possible minimum of the sum  $U + V$  is also achieved by the same TMSS located at  $(\sqrt{u^2 - v^2}, \sqrt{u^2 - v^2})$ .

The fact that the product condition is better than the sum condition is generally stressed in [45, 46] and the results of Refs. [10, 45, 46] essentially include the condition of Eq. (10) although the EPR-like operators are not normalized so that their partial transpositions form a pair of the canonical variables. For the case of  $|u| = |v|$ , the separable condition of Eq. (10) is derived in [41, 43, 44]. From the superiority of the product condition and the fact [6, 8] that any Gaussian entangled state can be detected by the violation of the sum condition, it is concluded [45] that any two-mode Gaussian entangled state can also be detected by the violation of the product condition. There is an approach to consider that the sum condition is a condition for a quadratic Hamiltonian, thereby a separable condition on the variance of the normalized Hamiltonian is derived [48].

To estimate the left-hand sides of Eqs. (10) and (13) in the experiments, one may perform the joint quadrature measurement of  $\hat{x}_A \hat{x}_B$  and  $\hat{p}_A \hat{p}_B$  (For the estimation of the covariance matrix of the two-mode state, it requires the measurement of  $\hat{x}_A \hat{p}_B$  and  $\hat{p}_A \hat{x}_B$ , additionally). The measured homodyne statistics of  $\hat{x}_A \hat{x}_B$  and  $\hat{p}_A \hat{p}_B$  give the six variances  $\{\langle \Delta^2 \hat{x}_A \rangle, \langle \Delta^2 \hat{x}_B \rangle, \langle \Delta^2 \hat{p}_A \rangle, \langle \Delta^2 \hat{p}_B \rangle, \langle \Delta \hat{x}_A \Delta \hat{x}_B \rangle, \langle \Delta \hat{p}_A \Delta \hat{p}_B \rangle\}$ . Then, the left-hand side of Eq. (10) can be determined for any given set of  $(u, v)$ . In practice, it is better to use the set of the parameters  $(u, v)$  so that the left-hand side of Eq. (10) becomes as small as possible. The minimum can be readily found by setting  $(u, v) = (\cos \theta, \sin \theta)$  and plotting the left-hand side of Eq. (10) as a function of  $\theta$ . The effect of this variable corresponds to a global rotation by the beamsplitter transformation. Although the joint squeezing  $\hat{S}_A \otimes \hat{S}_B$  to cover the separable boundary belongs to the set of local operations, it is not easy to access experimentally. In turn, to achieve the boundary of the product condition, the sum condition requires additional local squeezing operations. This suggests actual experimental advantage to use the product condition in place of the sum condition.

Note that the left-hand side of Eq. (13) becomes a quadratic form of  $\vec{u} := (u, v)^t$  as  $\vec{u}^t (M_x + M_p) \vec{u}$  where

$$\begin{aligned} M_x &:= \begin{pmatrix} \langle \Delta^2 \hat{x}_A \rangle & -\langle \Delta \hat{x}_A \Delta \hat{x}_B \rangle \\ -\langle \Delta \hat{x}_A \Delta \hat{x}_B \rangle & \langle \Delta^2 \hat{x}_B \rangle \end{pmatrix} \\ M_p &:= \begin{pmatrix} \langle \Delta^2 \hat{p}_A \rangle & \langle \Delta \hat{p}_A \Delta \hat{p}_B \rangle \\ \langle \Delta \hat{p}_A \Delta \hat{p}_B \rangle & \langle \Delta^2 \hat{p}_B \rangle \end{pmatrix}. \end{aligned} \quad (20)$$

Hence, the minimum of the left-hand side of Eq. (13) is given by the minimum eigenvalue of the matrix  $M_x + M_y$ . The minimum plays an important role in the analysis of Refs. [15, 47]. By using the matrices of Eq. (20), the left-hand sides of Eq. (10) can be expressed in a compact form  $(\vec{u}^t M_x \vec{u})(\vec{u}^t M_y \vec{u})$ .

#### IV. SEPARABLE CONDITION WITH THE COHERENT-STATE-BASED EPR-LIKE CORRELATION

In this section we use the partial transposition for the limitation on the  $\lambda$ -localization of Eq. (7), and derive the overlap separable condition corresponding to Eq. (5). It determines the strength of the EPR-like correlation in a coherent-state basis in order that the state is entangled. The maximal correlation on this basis is also obtained by a TMSS.

Let us consider the following positive operator

$$\begin{aligned} \hat{G}'_\lambda &:= \hat{R} \hat{G}_\lambda \otimes |0\rangle\langle 0| \hat{R}^\dagger \\ &= \int p_\lambda(\alpha) |v\alpha\rangle\langle v\alpha| \otimes |u\alpha\rangle\langle u\alpha| d^2\alpha, \end{aligned} \quad (21)$$

where the thermal state  $\hat{G}_\lambda$  is defined in Eq. (6) and the beamsplitter transformation  $\hat{R}$  is defined through its action on the coherent state  $\hat{R}|\alpha\rangle|0\rangle = |v\alpha\rangle|u\alpha\rangle$ . Since the spectrum of  $\hat{G}'_\lambda$  is the same as the spectrum of  $\hat{G}_\lambda$ , the physical limitation for the  $\lambda$ -localization of Eq. (7) also holds for  $\hat{G}'_\lambda$  as

$$\langle \hat{G}'_\lambda \rangle \leq \frac{\lambda}{1+\lambda}. \quad (22)$$

The equality is achieved by the product of the vacuum states  $|0, 0\rangle_{AB}$ .

From the partial transposition of  $\hat{G}'$  and the physical limitation of Eq. (22) we obtain the overlap separable condition [24]:

$$\langle \Gamma \hat{G}'_\lambda \rangle \leq \frac{\lambda}{1+\lambda}, \quad (23)$$

where the partial transposition of  $\hat{G}'$  can be written as

$$\Gamma(\hat{G}'_\lambda) = \int p_\lambda(\alpha) |v\alpha\rangle\langle v\alpha| \otimes |u\alpha^*\rangle\langle u\alpha^*| d^2\alpha. \quad (24)$$

Here, the action of the partial transposition map  $\Gamma$  of Eq. (11) induces the complex conjugation of the coherent-state amplitude of the second system in Eq. (21). The equality of Eq. (23) is also achieved by the product of the vacuum states  $|0, 0\rangle_{AB}$ . We can show that the overlap condition of Eq. (23) holds for any separable state as follows:

*Proof.*— For any separable state  $\rho_s$ ,  $\Gamma[\rho_s]$  is a density operator. Using Eq. (22) for  $\Gamma[\rho_s]$ , we have  $\langle \Gamma \hat{G}' \rangle_{\rho_s} = \text{Tr}(\rho_s \Gamma[\hat{G}']) = \text{Tr}(\Gamma[\rho_s] \hat{G}') \leq \lambda/(1+\lambda)$ . Hence, the violation of the condition in Eq. (23) implies that the state is entangled. ■

The expectation value of  $\Gamma \hat{G}'_\lambda$  in Eq. (24) is a weighted sum of the probability that the pair of coherent states

$|\alpha\rangle_A |g\alpha^*\rangle_B$  is contained in the given state, where  $g = u/v$  is a real number. Recalling that the complex amplitude is defined as  $\alpha = x + ip$ , the state overlap  $\langle \Gamma \hat{G}'_\lambda \rangle$  essentially represents the strength of the EPR-like correlation so that the relations  $x_A = gx_B$  and  $p_A = -gp_B$  hold, simultaneously. Actually the two relations can be combined to the single expression  $\alpha_A = g\alpha_B^*$  (See also FIG. 3). A different expression of  $\langle \Gamma \hat{G}'_\lambda \rangle$  in terms of the EPR-like uncertainties and its link to the separable conditions with the EPR-like uncertainties can be found in Sec. V. The relation of Eq. (23) to the original form of Eq. (5) is shown in Sec. VIB.

A violation of the overlap separable condition of Eq. (23) can be observed for the TMSS. From Eq. (19), the strength of the coherent-state-based EPR-like correlation  $\langle \Gamma \hat{G}' \rangle$  in Eq. (24) for the TMSS is calculated to be

$$\langle \Gamma(\hat{G}'_\lambda) \rangle_{\psi_\zeta} = \frac{\lambda(1-\zeta^2)}{u^2[1+\lambda/u^2-\zeta^2+(v/u-\zeta)^2]}. \quad (25)$$

If we set  $\sqrt{2}u = \sqrt{2}v = 1$  and  $2\zeta = 1$ , the right-hand side of Eq. (25) becomes  $3\lambda/(2+4\lambda)$ . Hence, for  $\lambda < 1$ , we can observe  $\langle \Gamma(\hat{G}'_\lambda) \rangle_{\psi_\zeta} > \lambda/(1+\lambda)$ . If the transformation  $\Gamma$  preserved the phase-space localization, this expression would imply the violation of the physical limitation for the  $\lambda$ -localization of Eq. (22) as a sort of the EPR paradox. In reality, the partial transposition  $\Gamma$  is not a physical transformation and it is no need to consider that the violation of Eq. (23) violates the physical limitation of Eq. (22). The paradox, in which entangled states can be “localized” beyond the limit achieved by the pair of coherent states, is essentially identical to the phenomenon that the EPR-like uncertainty product violates the canonical uncertainty limit discussed in the previous section. It might be helpful to consider that the phenomenon comes from the use of a strange way to sum the phase-space volume based on the partial transposed unit of the volume measure, in which the sign of the local momentum is inverted  $p_B \rightarrow -p_B$ . This inversion suggests the complex conjugation  $i \rightarrow -i$  because the sign of the commutation relation is changed due to the replacement  $[\hat{x}, \hat{p}] \rightarrow -[\hat{x}, \hat{p}]$ . Such a replacement affects the coherence between the two systems and some of entangled states exhibit seemingly abnormal phase-space volume. While the state overlap and the uncertainty product can be associated with the phase-space volume, it seems difficult to consider such an physical object in the case of the sum uncertainties.

Note that  $\Gamma \hat{G}'_\lambda$  of Eq. (24), as a density operator, is located at the point  $(U, V) = (1, 1)$  on the separable boundary of the  $U$ - $V$  plane as the vacuum state  $|0\rangle|0\rangle$  is located at the same point (See FIG. 2). Moreover, the state obtained by applying the collective local squeezing both on A and B to  $\Gamma \hat{G}'_\lambda$ , i.e.,  $\hat{S}_A \otimes \hat{S}_B \Gamma(\hat{G}'_\lambda) \hat{S}_A^\dagger \otimes \hat{S}_B^\dagger$  moves along the local minimum uncertainty boundary of Eq. (16) as  $\hat{S}_A \otimes \hat{S}_B |0, 0\rangle_{AB}$  does. We can see that  $\hat{S}_A \otimes \hat{S}_B \Gamma(\hat{G}'_\lambda) \hat{S}_A^\dagger \otimes \hat{S}_B^\dagger$  reduces to the product of the pure squeezed states  $\hat{S}_A \otimes \hat{S}_B |0, 0\rangle_{AB}$  in the pure limit  $\lambda \rightarrow \infty$ . Although the form of the mixture shows the correlation explicitly, its EPR-like correlation is no stronger than the correlation given by the uncorrelated product state  $|0, 0\rangle$  when it is measured either by the state overlap in Eq. (23) or by the product of the uncertainties in Eq. (10).

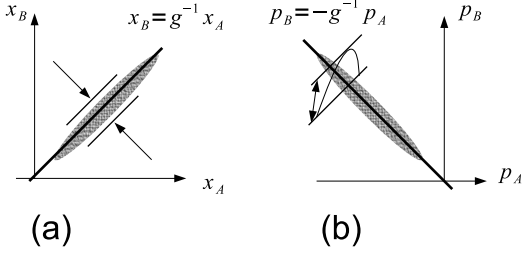


FIG. 3: A pair of the EPR-like particles exhibits a strong positive correlation on their positions  $x_A$  and  $x_B$ . It also exhibits a strong negative correlation on their momentums  $p_A$  and  $p_B$ . A total deviation from the lines of  $x_B = g^{-1}x_A$  and  $p_B = -g^{-1}p_A$  represents the strength of the EPR-like correlation. The deviation can be directly related to the width of the correlation distribution as (a). The strength of the correlation can also be related to the intensity of the distribution as (b). The two conditions of  $x_B = g^{-1}x_A$  and  $p_B = -g^{-1}p_A$  are combined into the single expression  $\alpha_B^* = g^{-1}\alpha_A$  with the complex amplitudes  $\alpha_A = x_A + ip_A$  and  $\alpha_B = x_B + ip_B$ . This suggests another form of the EPR-like correlation in terms of the distribution intensities associated with the pairs of coherent states  $\{|\alpha\rangle_A |g\alpha^*\rangle_B\}$ .

As was mentioned above, the strength of the coherent-state-based EPR-like correlation  $\langle \Gamma \hat{G}'_\lambda \rangle$  is a state overlap to the classically correlated state. It simply suggests the probability that the state contains the conjugate coherent-state pairs, and the separable condition of Eq. (23) gives the threshold of the pair appearance in order that the state is entangled. The maximum of the coherent-state-based EPR-like correlation  $\langle \Gamma(\hat{G}'_\lambda) \rangle$  is given by the operator norm of  $\Gamma(\hat{G}'_\lambda)$  as

$$\begin{aligned} \max_\rho \langle \Gamma(\hat{G}'_\lambda) \rangle_\rho &= \|\Gamma(\hat{G}'_\lambda)\| \\ &= \frac{4}{(\nu_+ + 1)(\nu_- + 1)} \\ &= \frac{2\lambda}{1 + \lambda + \sqrt{(1 + \lambda)^2 - 4u^2v^2}} \end{aligned} \quad (26)$$

where we use the symplectic eigenvalues  $\nu_\pm$  of  $\Gamma(\hat{G}'_\lambda)$  defined in Eq. (35). This maximum value is achieved by the TMSS of Eq. (19) when we set

$$\zeta = [(1 + \lambda) - \sqrt{(1 + \lambda)^2 - 4u^2v^2}]/(uv). \quad (27)$$

Hence, the EPR-like correlation can be maximized by the TMSS on the coherent-state basis as well as on the basis of the quadrature variances. From this fact and the fact that  $\Gamma \hat{G}'_\lambda$  is located at the separable boundary of the  $U$ - $V$  plane in FIG. 2, it might be feasible to consider that  $\langle \Gamma \hat{G}'_\lambda \rangle - \lambda/(1 + \lambda)$  is measuring a sort of distance from the point on the separable boundary  $(U, V) = (1, 1)$  so that  $\langle \Gamma \hat{G}'_\lambda \rangle - \lambda/(1 + \lambda)$  is maximized for the TMSS located on the physically possible boundary and vanishes at the separable boundary. In a strict sense, the separable boundary and the boundary given by  $\langle \Gamma \hat{G}'_\lambda \rangle = \lambda/(1 + \lambda)$  are different, at least for the Gaussian states (See, Sec. V).

In general, it is not necessary to choose the symmetric Gaussian distribution to discuss the localization, and we

can proceed similar discussion with a different distribution. For any positive operator with a positive- $P$  representation  $\rho_{pp}$  on a single mode, a two-mode operator  $\hat{R}\rho_{pp} \otimes |0\rangle\langle 0| \hat{R}^\dagger$  is an unnormalized separable state and the following relation holds  $\langle \hat{R}\rho_{pp} \otimes |0\rangle\langle 0| \hat{R}^\dagger \rangle \leq \|\rho_{pp}\|$  since  $\langle \rho_{pp} \rangle \leq \|\rho_{pp}\|$ . By taking the partial transposition we have a separable condition

$$\langle \Gamma[\hat{R}\rho_{pp} \otimes |0\rangle\langle 0| \hat{R}^\dagger] \rangle \leq \|\rho_{pp}\|. \quad (28)$$

If we know the  $P$  representation  $\rho_{pp} = \int P(\alpha)|\alpha\rangle\langle\alpha|d^2\alpha$ , the partial transposition can be calculated as  $\Gamma[\hat{R}\rho_{pp} \otimes |0\rangle\langle 0| \hat{R}^\dagger] = \int P(\alpha)|v\alpha\rangle\langle v\alpha| \otimes |u\alpha^*\rangle\langle u\alpha^*|d^2\alpha$ . The condition of Eq. (28) with a non-Gaussian distribution of  $P(\alpha)$  might be useful when the expectation value  $\langle |u\alpha\rangle\langle u\alpha| \otimes |v\alpha\rangle\langle v\alpha| \rangle$  is obtained for a limited number of the amplitude  $\alpha$  in the real experiments. In such a case, one can choose  $P(\alpha)$  as a discrete distribution associated with the observed set of the amplitudes. Further, analysis of potential utilities of this approach beyond the case of the symmetric Gaussian distribution is left for future works.

## V. EPR-LIKE CORRELATION FOR THE DETECTION OF TWO-MODE GAUSSIAN ENTANGLEMENT

In this section we apply the overlap condition of Eq. (23) for the two-mode Gaussian states in a standard form of the covariance matrix. In the flat-distribution limit ( $\lambda \rightarrow 0$ ), the overlap condition can be described by the normalized EPR-like uncertainties similar to the cases of the sum condition and the product condition. An embrace relation on these separable conditions suggests the utility of the coherent-state-based approach for the class of two-mode Gaussian states.

Let us consider the covariance matrix of a two-mode state  $\rho$

$$\gamma_\rho := \langle \Delta \hat{d} \Delta \hat{d}^t + (\Delta \hat{d} \Delta \hat{d}^t)^t \rangle_\rho \quad (29)$$

where  $\hat{d} := (\hat{x}_A, \hat{p}_A, \hat{x}_B, \hat{p}_B)^t$ . The physical requirement for the covariance matrix is given by  $\gamma + i\Omega \geq 0$  where

$$\Omega := \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} = J \oplus J, \quad J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (30)$$

with the normalization  $[\hat{x}_A, \hat{p}_A] = i$  and  $[\hat{x}_B, \hat{p}_B] = i$  (We set  $C = ||[\hat{x}, \hat{p}]/2 = 1/2$  henceforth). The characteristic function of a two-mode density operator  $\rho$  is defined by

$$\chi(\xi) := \text{Tr}[\rho \exp(i\hat{d}^t \xi)] \quad (31)$$

where  $\xi = (\xi_1, \xi_2, \xi_3, \xi_4)^t \in \mathbb{R}^4$  is a real vector. The density operator  $\rho$  can be written by the inverse of the Fourier transform as

$$\rho = (2\pi)^{-2} \int_{\mathbb{R}^4} \chi(\xi) \exp(-i\hat{d}^t \xi) d\xi. \quad (32)$$

We call the state is a Gaussian state if its characteristic function is Gaussian as

$$\chi(\xi) = \exp(i\hat{d}^t \xi - \frac{1}{4}\xi^t \gamma \xi), \quad (33)$$

where  $d = \langle \hat{d} \rangle = \text{Tr}[\rho \hat{d}]$  is the mean of the phase-space position. The two-mode Gaussian state is completely characterized by its covariance matrix  $\gamma$  and the mean  $d$ . The mean  $d$  can be freely chosen by applying local displacement operators, and is irrelevant to the inseparability. Hence, we consider the zero-mean case  $d = 0$  in the following discussion.

The operator  $\Gamma(\hat{G}'_\lambda)$  of Eq. (24) is a density operator of a Gaussian state and its covariance matrix is given by

$$\gamma_0 = I_4 + \frac{2}{\lambda} \begin{pmatrix} v^2 I_2 & uvZ \\ uvZ & u^2 I_2 \end{pmatrix}, \quad (34)$$

where  $I_4 = \text{diag}(1, 1, 1, 1)$ ,  $I_2 = \text{diag}(1, 1)$ , and  $Z = \text{diag}(1, -1)$ . The symplectic eigenvalues [4, 49, 50] of  $\gamma_0$  is given by

$$\nu_{\pm} = \sqrt{(1 + \lambda)^2 - 4u^2v^2} \pm (u^2 - v^2). \quad (35)$$

From Eqs. (32) and (33), for two Gaussian states  $\rho$  and  $\sigma$  with the zero means, their overlap can be expressed in terms of their covariance matrices as

$$\text{Tr}(\rho\sigma) = \left[ \det \left( \frac{\gamma_\rho + \gamma_\sigma}{2} \right) \right]^{-1/2}. \quad (36)$$

From this relation, the expectation value of Eq. (24) for a Gaussian state  $\rho$  can be written as

$$\langle \Gamma(\hat{G}'_\lambda) \rangle = \text{Tr}_\rho \Gamma(\hat{G}'_\lambda) = \left[ \det \left( \frac{\gamma_\rho + \gamma_0}{2} \right) \right]^{-1/2}, \quad (37)$$

and the overlap separable condition of Eq. (23) turns out to be

$$\det \left( \frac{\gamma_\rho + \gamma_0}{2} \right) \geq 1 + \frac{2}{\lambda} + \frac{1}{\lambda^2}. \quad (38)$$

The left-hand side of this expression can be simpler when  $\gamma_\rho$  is in the direct-sum form similar to the  $\gamma_0$  of Eq. (34). It is always possible to transform the covariance matrix into the direct-sum structure within the local Gaussian transformation [6, 8]. We thus consider the covariance matrix with the following direct-sum form:

$$\gamma_\rho = \begin{pmatrix} n_1 & 0 & c_1 & 0 \\ 0 & n_2 & 0 & c_2 \\ c_1 & 0 & m_1 & 0 \\ 0 & c_2 & 0 & m_2 \end{pmatrix}. \quad (39)$$

Here, it is not necessary to consider an irreducible form with  $n_1 = n_2$  and  $m_1 = m_2$  as in [6, 8]. For the direct-sum form, the condition of Eq. (38) leads to

$$\begin{aligned} & \det \left( \frac{\gamma_\rho + \gamma_0}{2} \right) \\ &= \frac{1}{16} \left[ (n_1 + 1)(m_1 + 1) - c_1^2 + \frac{2}{\lambda} (u^2 n_1 - 2uvc_1 + v^2 m_1) \right] \\ & \quad \times \left[ (n_2 + 1)(m_2 + 1) - c_2^2 + \frac{2}{\lambda} (u^2 n_2 + 2uvc_2 + v^2 m_2) \right] \\ & \geq 1 + \frac{2}{\lambda} + \frac{1}{\lambda^2}. \end{aligned} \quad (40)$$

In the flat-distribution limit  $\lambda \rightarrow 0$ , the coefficients of  $\lambda^{-2}$  are left and we thus have the following condition:

$$\begin{aligned} & \frac{1}{4} (u^2 n_1 + v^2 m_1 - 2c_1 uv + 1) \\ & \times (u^2 n_2 + v^2 m_2 + 2c_2 uv + 1) \geq 1. \end{aligned} \quad (41)$$

This condition is simply expressed in terms of the EPR-like uncertainties as

$$\frac{1}{4} (U + 1)(V + 1) \geq 1. \quad (42)$$

where we use the relations from Eq. (15) with  $C = 1/2$ ,

$$\begin{aligned} U &= 2\langle \Delta^2 (u\hat{x}_A - v\hat{x}_B) \rangle = (u^2 n_1 + v^2 m_1 - 2uvc_1) \\ V &= 2\langle \Delta^2 (u\hat{p}_A + v\hat{p}_B) \rangle = (u^2 n_2 + v^2 m_2 + 2uvc_2). \end{aligned} \quad (43)$$

Therefore, a bit surprisingly, it turns out that the overlap condition can be seen as a separable condition with the EPR-like uncertainties.

Taking square root on both sides of Eq. (42) and using the relation  $U + V + 2 \geq 2\sqrt{U+1}\sqrt{V+1}$  we can reproduce Eq. (17), i.e., we have  $U + V \geq 2$  again. This is the main result of this section. We have found the fact that the sum condition is generated from the overlap condition similar to the case that the sum condition of Eq. (13) is generated from the product condition of Eq. (10). This result implies that, for the class of Gaussian states expressed in the standard form, the overlap condition is tighter than the sum condition. This shows that the overlap condition has a substantial utility to detect the Gaussian entanglement as well as the conditions based on the standard EPR-like correlation of Eq. (2). To be specific, we have shown that detectable Gaussian entangled states covered by the sum condition are also covered by the overlap condition provided that the state is in the standard form of Eq. (39). On the other hand, it is known [6] that any two-mode entangled Gaussian state can be detectable by the sum condition by using a specific standard form also written in the form of Eq. (39). Therefore, it is concluded that the overlap condition is capable of detecting the set of two-mode Gaussian entangled states.

We can also show that the product condition of Eq. (16) can reproduce the overlap condition of Eq. (42) as follows. From Eq. (16) we have  $UV + 2\sqrt{UV} + 1 \geq 4$ . From this relation and  $U + V \geq 2\sqrt{UV}$  we have  $UV + U + V + 1 \geq 4$ . This relation is nothing but Eq. (42). We thus have proven the embrace relation for the three separable conditions:

$$\text{Eq. (16)} \subset \text{Eq. (42)} \subset \text{Eq. (17)}. \quad (44)$$

Now, in terms of the EPR-like uncertainties  $U$  and  $V$ , the three separable conditions of Eqs. (10), (13), and (23) are simply expressed in Eqs. (16), (17), and (42), respectively. Their embrace relation of Eq. (44) is displayed on the  $U$ - $V$  plane in FIG. 4. Thereby, geometrically we can prove that the overlap condition is tighter than the sum condition and that the product condition (10) is tighter than the overlap condition. It has been already mentioned in Sec. III that any point of the boundary of the product condition is achieved by the separable states and the product condition gives the tight threshold of the standard EPR-like correlation in order that the state is entangled. These facts would deepen the understanding of the entanglement of two-mode Gaussian states, and the interrelations between the three separable conditions would be of fundamental to comprehend the relation between the EPR-like correlation and entanglement.



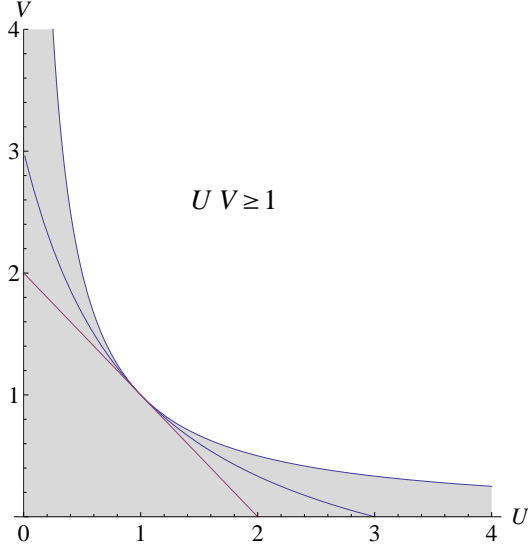


FIG. 4: (Color online) The three separable conditions for the Gaussian states are described by the three curves on the  $U$ - $V$  plane associated with the variances for the EPR-like operators ( $U, V$ ). The three curves are inscribed at the single point  $(U, V) = (1, 1)$ . The product condition of Eq. (10) implies that any state is entangled if its location on the  $U$ - $V$  plane ( $U, V$ ) is below the inverse proportional curve  $UV = 1$  (gray regime). The boundary of the sum condition of Eq. (13) is given by the straight line  $U + V = 2$ . The boundary of the overlap condition of Eq. (23) for the Gaussian states in the standard form is given by  $(U + 1)(V + 1) = 4$ , which includes the three points  $(1, 1)$ ,  $(3, 0)$ , and  $(0, 3)$ . This curve is lying in the middle of the two lines. It shows that the performance of the overlap condition is in between the product condition and the sum condition.

## VI. EXPERIMENTAL MEASUREMENT SCHEMES AND RELATION TO THE QUANTUM-DOMAIN CONDITION

In this section we describe how to estimate the state overlap to the EPR-like correlated classical mixture  $\langle \Gamma G'_\lambda \rangle$  of Eq. (24) in experiments. We also consider an experimental quantum teleportation scheme and discuss the relation between the overlap separable condition of Eq. (5) [Eq. (23)] and the quantum-domain condition of Eq. (3).

### A. Measurement for the projection to the conjugate pairs of coherent states

The heterodyne measurement corresponds to a projection to the coherent state and its positive-operator-valued-measure elements are symbolically written as  $\{|\alpha\rangle\langle\alpha|/\pi\}$ . If we perform heterodyne measurement each of the two modes  $A$  and  $B$ , then we can obtain the joint probability distribution associated with the projection to the product of the coherent states  $|\alpha\rangle_A |\beta\rangle_B$  where  $\alpha$  and  $\beta$  correspond to the outcomes of the measurement on the system  $A$  and the system  $B$ , respectively. From this joint probability distribution, the strength of the coherent-state-based EPR-like correlation  $\langle \Gamma G'_\lambda \rangle$  for any pair of the parameters  $(u, v)$  can be calculated. This enables us

to check the overlap separable condition of Eq. (23) in principle. This may not be an efficient way for entanglement detection because the heterodyne statistics include the information of the full-tomographic reconstruction [51].

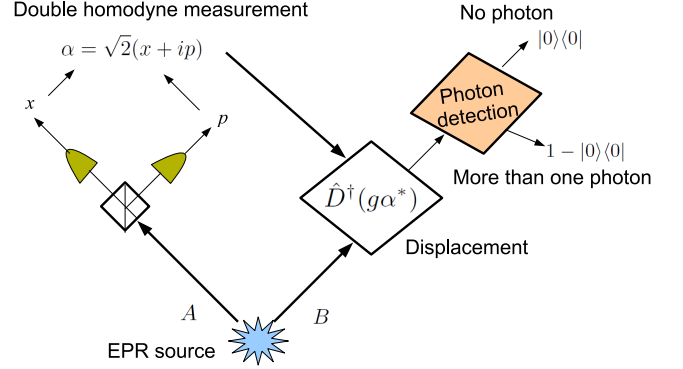


FIG. 5: (Color online) The projection probability to a pair of coherent states  $|\alpha\rangle |g\alpha^*\rangle$  can be measured by a heterodyne (double homodyne) measurement and a photon detection. The measurement outcome of the heterodyne measurement  $(x, p)$  determines the amplitude of the coherent state  $\alpha = \sqrt{2}(x + ip)$  of the mode  $A$ . The photon detection determines whether the number of the photon in the measured mode is zero or more than 1; It is the projection to one of the two subspaces  $|0\rangle\langle 0|$  and  $I - |0\rangle\langle 0|$  where  $I = \sum_{n=0}^{\infty} |n\rangle\langle n|$ . The photon detection of the mode  $B$  after the displacement operation  $\hat{D}^\dagger(g\alpha^*)$  gives the information whether or not the state is initially in the coherent state  $|g\alpha^*\rangle = \hat{D}(g\alpha^*)|0\rangle$  because this state becomes vacuum after the displacement as  $\hat{D}^\dagger(g\alpha^*)|g\alpha^*\rangle = |0\rangle$ .

In turn, if a pair of the parameters  $(u, v)$  is specified beforehand, we only need to consider the probability associated with the specific pairs of the states  $|\alpha\rangle_A |g\alpha^*\rangle_B$  with  $g = v/u$ . In order to measure this probability, a possible measurement process is composed of a heterodyne measurement and a photon detection with a feedforward control as in FIG. 5. We first perform the heterodyne measurement on the system  $A$ . Then, according to the outcome of the heterodyne measurement  $\alpha$ , we apply the displacement operation with an amount of the displacement  $g\alpha^*$  on the system  $B$ . Finally, we perform the photon detection of the system  $B$ . It confirms whether or not the system  $B$  was  $|g\alpha^*\rangle$ . This is because the displacement  $\hat{D}^\dagger(g\alpha^*)$  transforms the conjugate coherent state  $|g\alpha^*\rangle$  to the vacuum state as  $|0\rangle = \hat{D}^\dagger(g\alpha^*)|g\alpha^*\rangle$  and the vacuum state is correctly discriminated by the photon detection as the no-photon event. This measurement technique has been demonstrated recently [52–54]. Repeating this process we can obtain the probability that the pair state  $|\alpha\rangle_A |g\alpha^*\rangle_B$  is contained in the total system initially. From the measured expectation values  $\langle |\alpha\rangle\langle\alpha| \otimes |g\alpha^*\rangle\langle g\alpha^*| \rangle$ , a class of the separable conditions associated with Eq. (28) can be checked as well.

### B. Continuous-variable quantum teleportation and the quantum-domain condition

An interesting application of the EPR correlation can be found in the experiments of continuous-variable quan-

tum teleportation [36] and quantum memory [55] protocols. The average fidelity of quantum teleportation for coherent states is a widely accepted measure to estimate the performance of the state transfer [19, 20, 23]. It might be insightful to consider the relation between the teleportation fidelity and the EPR-like correlation on the coherent-state basis in the quantum teleportation process.

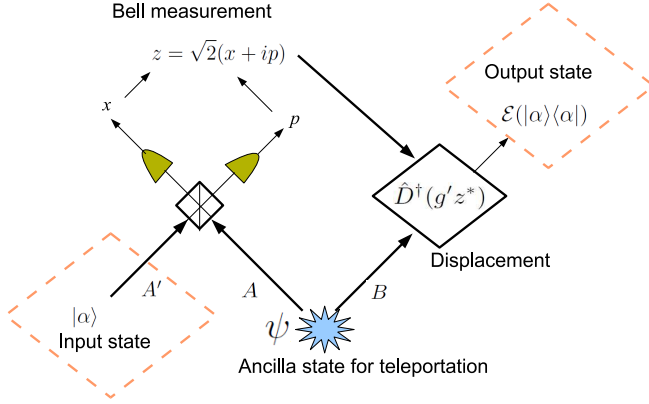


FIG. 6: (Color online) A continuous-variable quantum teleportation process  $\mathcal{E}$  of an input coherent state  $|\alpha\rangle$  from the mode  $A'$  to the mode  $B$  by using a possibly entangled state  $\psi$  on the modes  $A$  and  $B$ . The outcome  $z$  of the Bell measurement on the joint mode  $AA'$  is forwarded to make a displacement  $\hat{D}^\dagger(g'z^*)$  on the mode  $B$  where  $g' \geq 0$  is the teleportation gain.

Let us consider the quantum teleportation process [37] for a coherent state  $|\alpha\rangle$  from the system  $A'$  to the system  $B$  by using a possibly entangled state  $\psi$  on the joint system  $AB$  as in FIG. 6. The teleportation process consists of the two steps: (i) a Bell measurement of the joint system  $AA'$ , and (ii) a feedforward displacement operation on the system  $B$  associated with the Bell-measurement outcome  $(x, p)$ . The first step can be done by a half-beamsplitter transformation  $\hat{R}$  and two homodyne measurements. Suppose that the state of the system  $AA'$  is a product of coherent states  $|\gamma\rangle_A |\alpha\rangle_{A'}$ . Then, the joint probability density that the Bell-measurement outcome is  $(x, p)$  can be written as

$$\begin{aligned} & \left| \langle x |_A \langle p |_{A'} \hat{R} |\gamma\rangle_A |\alpha\rangle_{A'} \right|^2 \\ &= |\langle x | (\gamma - \alpha) / \sqrt{2} \rangle|^2 |\langle p | (\gamma + \alpha) / \sqrt{2} \rangle|^2 \\ &= \frac{2}{\pi} e^{-2 \left( x - \frac{\Re e[\gamma] - \Re e[\alpha]}{\sqrt{2}} \right)^2} e^{-2 \left( p - \frac{\Im m[\gamma] + \Im m[\alpha]}{\sqrt{2}} \right)^2} \\ &= \frac{2}{\pi} e^{-|\gamma - \alpha^* - z|^2} \\ &= \frac{2}{\pi} \left| \langle \alpha^* | \hat{D}^\dagger(z) |\gamma\rangle \right|^2 \end{aligned} \quad (45)$$

where  $\langle x |$  ( $\langle p |$ ) represents the eigenbra of the quadrature  $\hat{x}$  ( $\hat{p}$ ) belonging to the eigenvalue  $x$  ( $p$ ), the beam-splitter transformation is given by  $\hat{R}_{AA'} |\gamma\rangle_A |\alpha\rangle_{A'} = |(\gamma - \alpha) / \sqrt{2}\rangle_A |(\gamma + \alpha) / \sqrt{2}\rangle_{A'}$ , and the complex amplitude is defined by  $z := \sqrt{2}(x + ip)$ . With this measurement outcome  $z$  and a gain parameter  $g' \geq 0$ , the second step can be described by the displacement operation  $\hat{D}^\dagger(g'z^*)$  on the mode  $B$ . By using Eq. (45) and the  $P$

representation  $\psi_{BA} = \int d^2\beta d^2\gamma P(\beta, \gamma) |\beta\rangle\langle\beta|_B \otimes |\gamma\rangle\langle\gamma|_A$ , we can express the state after the teleportation process as

$$\begin{aligned} \mathcal{E}(|\alpha\rangle\langle\alpha|) &= \int dx dp \hat{D}_B^\dagger(g'z^*) \langle x |_A \langle p |_{A'} \hat{R}_{AA'} \\ &\quad \psi_{BA} \otimes |\alpha\rangle\langle\alpha|_{A'} \hat{R}_{AA'} |x\rangle_A |p\rangle_{A'} \hat{D}_B(g'z^*) \\ &= \int \frac{d^2z}{\pi} \hat{D}_B^\dagger(g'z^*) \langle \alpha^* |_A \hat{D}_A^\dagger(z) \\ &\quad \times \int d^2\beta d^2\gamma P(\beta, \gamma) |\beta\rangle\langle\beta|_B \otimes |\gamma\rangle\langle\gamma|_A \\ &\quad \times \hat{D}_A(z) |\alpha^*\rangle \hat{D}_B(g'z^*) \\ &= \langle \alpha^* | \psi'_{BA} | \alpha^* \rangle_A \end{aligned} \quad (46)$$

where we define

$$\psi'_{BA} = \int \frac{d^2z}{\pi} \hat{D}_A^\dagger(z) \hat{D}_B^\dagger(g'z^*) \psi_{BA} \hat{D}_B(g'z^*) \hat{D}_A(z). \quad (47)$$

Note that  $\psi'_{BA}$  is not a normalized density operator while  $\langle \alpha^* | \psi'_{BA} | \alpha^* \rangle_A$  is a normalized density operator on the system  $B$ .

By substituting Eq. (46) into the channel condition of Eq. (3) we can obtain the relation

$$\int d^2\alpha p_\lambda(\alpha) \langle \sqrt{\eta}\alpha | \langle \alpha^* | \psi'_{BA} | \alpha^* \rangle_A | \sqrt{\eta}\alpha \rangle_B \leq \frac{1 + \lambda}{1 + \lambda + \eta}. \quad (48)$$

This relation can be regarded as a condition on the two-mode “state,”  $\psi'_{BA}$ . On the other hand, by changing the variable of integration  $\alpha' = v\alpha$  and making the replacement  $\lambda/v^2 \rightarrow \lambda$ , the overlap separable condition of Eq. (23) can be transformed into the form of Eq. (5)

$$\left\langle \int p_\lambda(\alpha) |\alpha\rangle\langle\alpha| \otimes |\sqrt{\eta}\alpha^*\rangle\langle\sqrt{\eta}\alpha^*| d^2\alpha \right\rangle \leq \frac{\lambda}{1 + \lambda + \eta} \quad (49)$$

where we set  $\eta = \frac{u^2}{v^2} = \frac{1}{v^2} - 1$ . Note that this parameter  $\eta$  comes from the parameter  $a$  of the original form of the EPR-like correlation in Eq. (2) through the relations of Eq. (14) whereas it is introduced as a transmission parameter of the quantum channel in Refs. [23, 56].

Now, the same Gaussian weighted conjugate pairs of coherent states appear on both of the left-hand sides of Eqs. (48) and (49), and the coherent-state-based entanglement-breaking condition of Eq. (3) and the overlap separable condition of Eq. (23) are expressed in parallel forms. Actually, the measurement processes of FIG. 5 and the teleportation process of FIG. 6 are in parallel, and the success of the quantum teleportation and the inseparability based on the coherent-state pairs can be estimated by similar experimental setups.

Interestingly, we can find the form of the coherent-state-based EPR-like correlation in the left-hand side of Eq. (48), and the strength of the EPR-like correlation on the coherent-state basis is also related to the performance of the quantum teleportation. If we set the gain  $g' = \sqrt{\eta}$  we can rewrite the integrand in the left-hand side of Eq.

(48) as

$$\begin{aligned} & \langle \sqrt{\eta}\alpha | \langle \alpha^* | \psi'_{BA} | \alpha^* \rangle_A | \sqrt{\eta}\alpha \rangle_B \\ &= \int \frac{d^2 z}{\pi} \langle \sqrt{\eta}\beta | \langle \beta^* | \psi_{BA} | \beta^* \rangle_A | \sqrt{\eta}\beta \rangle_B, \end{aligned} \quad (50)$$

with  $\beta = \alpha + z^*$ . From this expression with a change of the integration variable  $d^2\beta = d^2\alpha$  we can perform the integration of  $z$  in the left-hand side of Eq. (48) as

$$\begin{aligned} & \int d^2\alpha p_\lambda(\alpha) \langle \sqrt{\eta}\alpha | \langle \alpha^* | \psi'_{BA} | \alpha^* \rangle_A | \sqrt{\eta}\alpha \rangle_B \\ &= \int d^2\beta \left( \int \frac{d^2 z}{\pi} p_\lambda(\beta - z^*) \right) \\ & \quad \times \langle \sqrt{\eta}\beta | \langle \beta^* | \psi_{BA} | \beta^* \rangle_A | \sqrt{\eta}\beta \rangle_B \\ &= \frac{1}{\pi} \int d^2\beta \langle \sqrt{\eta}\beta | \langle \beta^* | \psi_{BA} | \beta^* \rangle_A | \sqrt{\eta}\beta \rangle_B. \end{aligned} \quad (51)$$

This expression no more includes the parameter  $\lambda$ , and we have the following condition for the possible entangled state  $\psi$  by setting  $\lambda = 0$  in the right-hand side of Eq. (48):

$$\frac{1}{\pi} \int d^2\beta \langle \sqrt{\eta}\beta | \langle \beta^* | \psi_{BA} | \beta^* \rangle_A | \sqrt{\eta}\beta \rangle_B \leq \frac{1}{1+\eta}. \quad (52)$$

This condition corresponds to the condition of Eq. (49) with  $\lambda = 0$ . Therefore, we have started with the teleportation process and the channel condition of Eq. (3), and have finally derived the separable condition of Eq. (49) with  $\lambda = 0$  for the ancilla state of the teleportation. It is a converse direction of the derivation of the channel condition of Eq. (3) in Ref. [24] where the channel condition is derived by using Eq. (5).

Equation (51) or the left-hand side of Eq. (52) suggests that the teleportation fidelity is essentially determined by the coherent-state-based EPR-like correlation on the possible entangled ancilla state  $\psi$  in the teleportation process. To be specific, for any physical state  $\psi$  whose EPR-like correlation [the left-hand side of Eq. (52)] is larger than  $1/(1+\eta)$  [the right-hand side of Eq. (52)], there exists  $\lambda > 0$  to violate the condition of Eq. (3) provided that the teleportation process is ideal. Therefore, we have succeeded in reconciling the condition on the channel fidelity to the condition on the resource entanglement state for the teleportation not only for the unit-gain case but also for non-unit-gain cases, and it has turned out that the condition on the resource state required to non-classically transfer an unknown state is directly presented by a threshold on the strength of the EPR-like correlation. Note that the left-hand side of Eq. (52) includes an integration over the entire complex plane. A lower bound of this term can be estimated from experimentally observed correlations in a finite area of the complex plane, and this lower bound is supposed to exceed the threshold in the experimental entanglement verification, conservatively. Note also that Refs. [38, 39] are showing relationships between the fidelity and entanglement assuming specific forms on the possible entangled state. It might be interesting to revisit those relations associated with the present formulation.

As a short summary, we have observed that the EPR-like correlation in the coherent-state basis plays a central role in the following three objectives:

- Reconciling the effect of entanglement to the fidelity on the continuous-variable quantum teleportation.
- Detecting the inseparability of two-mode Gaussian states.
- Resolving the quantum benchmark problem on one-mode quantum channels [24].

We thus would like to conclude that the coherent-state-based EPR-like correlation is an essential tool to comprehend entanglement both in the canonical continuous-variable systems and in their dynamics.

## VII. CONCLUSION

We have introduced the notion of the coherent-state-based EPR-like correlation as a state overlap to a classically correlated coherent-state mixture. The separable threshold of this correlation was derived by using the partial transposition on a limitation of the phase-space localization. A parallel formulation was given for the product separable condition concerning the standard EPR-like correlation. The product condition describes an optimal separable boundary with respect to the pair of the EPR-like uncertainties and this boundary curve is covered by the product of the squeezed states. For the two-mode Gaussian states given in a standard direct-sum form of the covariance matrix, the overlap separable condition can be written in an accessible form with the EPR-like uncertainties, and a simple embrace relation between the overlap condition, the sum condition, and the product condition has been established. It implies an essential usefulness of the coherent-state-based approach. We have also addressed how to detect the state overlap experimentally by the heterodyne detection and the following photon detection with a feedforward control. The measurement scheme is similar to the scheme for continuous-variable quantum teleportation, and the coherent-state-based EPR-like correlation can be related to the teleportation fidelity for coherent states. Thereby, a parallelism between the overlap separable condition and the fidelity-based entanglement-breaking condition was pointed out. Moreover, we have succeeded in reconciling the condition on the channel fidelity to the condition on the resource entanglement state for the teleportation. This formulation includes the case of non-unit gain and it has turned out that the correlation on the resource state to be required for a genuine quantum teleportation is directly presented by the strength of the EPR-like correlation on the coherent-state basis. The results would offer a novel basis for further understanding the basic ingredients of quantum mechanics and quantum information science, such as, the EPR-like correlation, quantum teleportation, and continuous-variable entanglement, in a unified aspect.

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